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Expectations Equilibria with Dispersed Forecasts*

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A dispersion condition for traders' forecasts in a general equilibrium model with uncertainty and asymmetric information yields improved results for some (microeconomic) situations in which rational expectations equilibria need not exist. The hypothesis of suitably dispersed forecasts implies that consumers' aggregate excess demand is a continuous function and therefore a fixed point theorem may be applied to obtain a price vector (for each state of the world) such that markets clear. Stronger assumptions give existence of approximately rational expectations equilibria and the convergence of forecast distributions to rational expectations.

1. INTRODUCTION

For many markets under uncertainty, a large number of traders each act in a competitive manner. Preferences depend on the state of the world, and agents hold different beliefs about the meaning of observed endogenous variables, such as prices. This paper explores some of the consequences of perfectly competitive trading when agents have different beliefs. Perfect competition suggests the presence of many traders, each of whom is insignificant in the market. By different information, we mean that traders' forecasts—or assumed relationships between unknown parameters which affect preferences and observed prices in the market—are dispersed. Agents have truly different beliefs, although it might be the case that a high percentage of them have forecasts which are similar, or very close to the actual relationship. We shall analyze the possibilities for rational expectations equilibria in such a microeconomic model.

A prototype market having these characteristics is the stock market. At least a subset of traders is of large cardinality and consists of traders who

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individually play a quite minor role in determining market prices. Casual empiricism suggests that traders have different beliefs about future stock prices, and that each is quite firmly convinced that his beliefs are correct. In other words, agents' forecasts are dispersed and each agent admits no uncertainty about the validity of his own forecast.

The major results of this paper suggest that such markets may be quite well behaved, in spite of some counterexamples which indicate the opposite case. In particular, the existence of equilibrium price functions is assured when nonatomic traders have dispersed information. Moreover, we give examples of some conditions which imply that there are expectations equilibria in which agents' beliefs are approximately correct. In this class of examples, convergence of forecasts to the approximately rational expectations equilibria occurs as the traders learn about the market. These *approximately rational* expectations equilibria are not revealing in general.

By now, the possibility of nonexistence of rational expectations equilibria is a well-known phenomenon. Counterexamples to existence of equilibria when prices transmit information were first developed by Kreps [30]. "Open" counterexamples, in which nonexistence persists under sufficiently small perturbations of the economic data, have been discovered by Green [21] and by Jordan and Radner [29]. Recent positive results assert that the existence of strict rational expectations equilibria which are revealing is generic if the set of signals or states of the world is finite (Radner [35]) or of sufficiently low dimension relative to the number of commodities in the economy (Allen [2, 4]). Work by Jordan [28] suggests that "fewness" of states is essential for generic existence and revelation—in particular, if the set of states is of higher dimension than the price simplex, then, generically, the efficient markets hypothesis fails. However, if the set of states of the world is of *strictly* higher dimension than the price simplex, then, for a residual set of economies (satisfying slightly stronger assumptions than those used in [2, 4, 35]), there is a strict rational expectations equilibrium which is given by a price function (mapping from states of the world to price vectors) which is a two-to-one mapping and which is almost everywhere discontinuous, see [27]. Existence theorems have also been proved under special assumptions on the structure of demands or the distribution of the underlying random variable [12, 20, 22–25]. All of the literature discussed above has used models with finitely many traders. However, these results—and the relevant counterexamples—would also hold for type economies (economies with a continuum of traders in which only a finite number of types of traders are included).

Another approach is to abandon the requirement of *strict* rational expectations equilibria—that is, to allow agents to retain some doubts and thus to place positive conditional probability on events which are contradicted by their expectations. Radner [34] examined an imperfect markets model with finitely many traders. He showed that if the unconditional measure is absolutely continuous with respect to any agent's prior probability measure conditioned on any information, then an equilibrium price function (a measurable mapping from the set of states of the world into the appropriate price simplex) exists and is rational in the sense that agents' prior beliefs are never contradicted by their observation of the true state of the world after trading has occurred. The absolute continuity condition says that, no matter what is observed, each agent's prior beliefs, conditional on that information, are dispersed. No event which occurs with positive unconditional probability is ever assigned conditional probability zero by anyone, regardless of the available information. However, the major problem with Radner's "imperfect markets model" result is the lack of economic rationale for the dispersion assumption about conditional beliefs. If I see a blizzard from my window, or observe a price which occurs only when there is a blizzard, why should I assign positive probability to the event "sunshine, and temperature at least 75°F"? Perhaps this can be justified if agents are currently discovering the relationship between states of the world and prices, but, in that case, the learning process which yields such an "imperfect markets model" should be explicitly modeled formally. (This point was observed by Radner in [34].)

Perhaps this work could be viewed as an attempt to clarify the relationship between Radner's result on the existence of rational expectations equilibrium for imperfect market models and known results and counterexamples concerning the possibilities for existence of strict rational expectations equilibria. The key observation is the substitution of a condition about dispersion of information (or forecasts) among imperfectly informed agents for Radner's assumption (that no agent ever excludes as impossible any open set of prices). It is suggested that the dispersion condition is natural, compatible with the hypothesis that all agents strictly believe their information, and, to some extent, potentially testable.

The strategy used in this paper will be to examine expectations equilibria in a class of idealized perfectly competitive economies having many negligible agents. The philosophy behind the use of such models is that price taking behavior can be justified rigorously only in economies with a nonatomic continuum of agents. Moreover, a typical negligible agent need not worry that his own actions will reveal private information through market prices; thus the positive disincentive to utilize one's own information is removed. However, this would be a rather trivial modification of the basic model if it were not for the folk theorem that large economies tend to be better behaved than small economies. (See [8–10, 26], and for economies under uncertainty, [15, 16, 38].) This paper specializes this observation to economies in which prices convey information and attempts to apply "large economies" techniques to the problem of the existence of rational expectations equilibria. I believe that these techniques help us to understand the nature of the difficulty which is the source of nonexistence phenomena. In particular, they help to separate the discontinuity (which negates the use of fixed point theorems to prove existence) from the "greased pig" factor[1] (that in dynamic learning situations, expectations and prices may not converge).

In may ways, the real purpose of this paper is to explore a sensible modification of the rational expectations hypotheses. My interest is in precisely those cases which are not covered by the generic existence results [2, 4, 27, 35] for strict rational expectations equilibria. The nature of some of the assumptions used to obtain approximate rationality and convergence indicate that this approach tends to be less than satisfactory for general equilibrium analysis, although the assumptions compare favorably to those used in the partial equilibrium rational expectations literature (i.e., [12, 13, 22, 23]).

2. The Model

The model attempts to capture the phenomena of many negligible agents and dispersed forecasts among imperfectly informed agents. In order to make the dependence of endogenous variables on the state of the world nontrivial, I hypothesize that a positive fraction of agents have perfect information about the state of the world and that this knowledge is reflected in their demand functions through the maximization of a state-dependent utility function. For technical reasons, I shall work with well-behaved smooth functions throughout the model.

There is a continuum of imperfectly informed agents. A typical such agent will be denoted α. For simplicity, I will give these agents names in A, a closed subset of the unit interval[2] $[0, 1]$. The σ-field of measurable subsets of these agents will be precisely the Borel subsets $\mathscr{B}(A)$ of A. The measure describing the "weight" of a coalition of these agents is taken to be Lebesgue measure λ on $(A, \mathscr{B}(A))$. Then our nonatomic continuum of imperfectly informed agents is described by the measure space $(A, \mathscr{B}(A), \lambda)$.

Agents who are perfectly informed will be the subset $B = [0, 1] \backslash A$ of the unit interval. A typical such trader will be denoted β. For simplicity, assume that $A = [0, c]$ and $B = (c, 1]$, where $0 < c < 1$. Assume, also to simplify, that there is a nonatomic continuum of perfectly informed agents

[1] I owe this analogy to Bruce Greenwald.

[2] Both standard representations and distributions of agents' characteristics will be used for convenience. Note that any distribution on a separable complete metric space has a standard representation [26, p.50].

described by the measure space $(B, \mathscr{B}(B), \lambda)$. Then $c=\lambda(A)=\int_0^c dt$ and $1-c=\lambda(B)=\int_c^1 dt$ are the (strictly positive) fractions of imperfectly and perfectly informed agents respectively.

The set Ω of states of the world is a compact smooth m-dimensional manifold which is a subset of $\mathbb{R}^m$. Events, or measurable subsets of states of the world, consist of the Borel subsets $\mathscr{B}(\Omega)=\mathscr{F}$ of Ω. Let μ be a non-atomic probability measure on $(\Omega, \mathscr{F})$; it describes the objective or agreed-upon subjective probabilities of the occurrence of various events. The symbol ω will represent a typical state of the world. Assume that μ has a density g with respect to m-dimensional Lebesgue measure on Ω.

Consumption sets for all consumers are taken to be the strictly positive orthant $\mathbb{R}^l_{++}$ of l-dimensional Euclidean space, where l is the number (finite) of commodities present in the economy. Prices shall be normalized to lie in the $l-1$ dimensional open unit simplex in $\mathbb{R}^l$, $\Delta=\{q\in\mathbb{R}^l_{++} \mid \sum_{j=1}^l q_j=1\}$. Each agent α has a strictly positive initial endowment vector e_α which is independent of the state of the world, while for an agent β, his initial endowment function

$$e_\beta\colon (\Omega, \mathscr{F}) \to (\mathbb{R}^l_{++}, \mathscr{B}(\mathbb{R}^l_{++}))$$

is a (vector-valued) random variable (measurable function) which is, in fact, a C^1 mapping from Ω to $\mathbb{R}^l_{++}$. Assume that there is some compact subset K of $\mathbb{R}^l_{++}$ such that (i) for all $\alpha\in A$, $e_\alpha\in K$ and (ii) for all $\beta\in B$ and all $\omega\in\Omega$, $e_\beta(\omega)\in K$. This will have the effect of giving a uniform bound for budget sets and demands whenever prices are bounded away from zero. In addition, assume that initial endowments are measurable (in α and β).

To describe the dependence of agents' preferences on the state of the world, they shall be represented by state-dependent (cardinal) utility functions. Smoothness and measurability will be assumed. Define mappings

$$u\colon A\times\mathbb{R}^l_{++}\times\Omega\to\mathbb{R}$$

$$v\colon B\times\mathbb{R}^l_{++}\times\Omega\to\mathbb{R}$$

which are measurable in A or B, C^2 functions on $\mathbb{R}^l_{++}$ and C^1 functions on Ω (and hence jointly measurable). The evaluation $u_\alpha(x,\omega)$ shall mean the utility to agent α of consuming commodity bundle $x\in\mathbb{R}^l_{++}$ when the state of the world is ω, and $v_\beta(x,\omega)$ the analogous utility for agent β. For each $\alpha\in A$, $\beta\in B$, and $\omega\in\Omega$, assume the following:

(i) strict (differentiable) monotonicity: for each $x\in\mathbb{R}^l_{++}$, $D_x u_\alpha(x,\omega)\gg 0$ and $D_x u_\beta(x,\omega)\gg 0$.

(ii) strict (differentiable) concavity: for each $x\in\mathbb{R}^l_{++}$, $D_{xx}u_\alpha(x,\omega)$ and $D_{xx}v_\beta(x,\omega)$ are negative definite.

(iii) boundary condition: for all $\gamma \in \mathbb{R}$, $\mathrm{cl}_{\mathbb{R}^l}\{x \in \mathbb{R}^l_{++} \mid u_\alpha(x, \omega) = \gamma\}$ and $\mathrm{cl}_{\mathbb{R}^l}\{x \in \mathbb{R}^l_{++} \mid v_\beta(x, \omega) = \gamma\}$ are contained in $\mathbb{R}^l_{++}$.

The set of utilities satisfying (i), (ii), and (iii) is a G_δ subset of $C^2(\mathbb{R}^l_{++}, \mathbb{R})$, endowed with topology of C^2 uniform convergence on compact subsets [1, Theorem 2.1]. With these assumptions, for $\omega \in \Omega$ known to all agents, demands are C^1 functions on $\Omega \times \Delta$ which are measurable in α or β and uniformly bounded on compact subsets of Δ. They also satisfy the boundary condition that as $q_n \to q \in \partial\Delta$, demand (for each $\omega \in \Omega$, $\alpha \in A$ and $\beta \in B$) is unbounded.

Finally, we will describe agents' information. Each trader $\beta \in B$ is perfectly informed—he knows the true state of the world ω before trading, and thus his demand $x \in \mathbb{R}^l_{++}$ is chosen to maximize $v_\beta(\cdot, \omega)$ subject to $q \cdot x \leqslant q \cdot e_\beta(\omega)$, where $q \in \Delta$ is the price vector prevailing in the market. Note that the budget set is known with certainty, and, because these traders are assumed to be perfectly informed initially, their initial endowments do not tell them any additional information. The situation for the imperfectly informed traders $\alpha \in A$ is quite different. Their knowledge is summarized by forecast functions which relate the state of the world to the endogenous economic variables which they observe. Let

$$p_\alpha \colon \Omega \to \Delta$$

represent trader α's forecast or economic model. The p_α are assumed to be C^1 functions, and

$$\tilde{p} \colon (A, \mathscr{B}(A)) \to (C^1(\Omega, \bar{\Delta}), \mathscr{B}(C^1(\Omega, \bar{\Delta})))$$

is assumed to be a measurable function. The topology on $C^1(\Omega, \bar{\Delta})$ is given by the C^1 sup norm:

$$\| p_\alpha - p'_\alpha \| = \sup_{\omega \in \Omega} \{ \| p_\alpha(\omega) - p'_\alpha(\omega) \| + \| D_\omega p_\alpha(\omega) - D_\omega p'_\alpha(\omega) \| \}.$$

With this norm, $C^1(\Omega, \bar{\Delta})$ is a complete separable normed space. When agent α observes price vector $q \in \bar{\Delta}$, the agent maximizes expected utility conditioned on the event $\{\omega \in \Omega \mid p_\alpha(\omega) = q\}$ if there is some state of the world $\omega \in \Omega$ with $p_\alpha(\omega) = q$. On the other hand, if for all $\omega \in \Omega$, $p_\alpha(\omega) \neq q$, then agent α maximizes the unconditional expected utility $\int u_\alpha(\cdot, \omega)\, d\mu(\omega)$. In both cases, the budget set is $\{x \in \mathbb{R}^l_{++} \mid q \cdot x \leqslant q \cdot e_\alpha\}$, which is independent of $\omega \in \Omega$. We make the following assumption about the distribution of forecasts:

Dispersion Hypothesis. The marginal distributions $\lambda \circ \tilde{p}(\omega)^{-1}$ of $\lambda \circ \tilde{p}^{-1}$ have bounded densities with respect to $(l-1)$-dimensional

Lebesgue measure on Δ which are uniformly bounded in $\omega \in \Omega$. (Note that $\lambda \circ \tilde{p}^{-1}$ is a probability measure on $C^1(\Omega, \Delta)$; $\lambda \circ \tilde{p}(\omega)^{-1}$ is a probability measure on Δ for each ω.)

This says that, for any $\omega \in \Omega$, not too many of the imperfectly informed traders agree too much about the price vector which corresponds to that particular state of the world ω. For example, if the marginals were uniform on Δ for each $\omega \in \Omega$, the dispersion hypothesis would be satisfied. It would be violated if, for some $\omega \in \Omega$, the support of $\lambda \circ \tilde{p}(\omega)^{-1}$ was contained in a subset of Δ of strictly lower dimension. Other counterexamples would be provided by cases in which the distribution of $\lambda \circ \tilde{p}(\omega)^{-1}$ has an atom for some ω, or mass piles up in such a way as to "approach" a marginal distribution with atoms.

Remarks. All of our results would hold if the measure space of perfectly informed agents contained atoms. However, the nonatomicity assumption simplifies notation and is consistent with our interpretation of many negligible traders.

3. Continuity of Demand and the Existence of a Market-Clearing Price Function

I shall prove that, under the dispersion hypothesis, demand in each state of the world is a continuous function of price and that there exists a market-clearing price function $p: \Omega \to \Delta$. To fix notation, write $EU_\alpha(x; p_\alpha, q)$ for α's expected utility of consuming $x \in \mathbb{R}^l_{++}$, given forecast $p_\alpha \in C^1(\Omega, \bar{\Delta})$, when $q \in \Delta$ is the prevailing price vector. Assume that the Dispersion Hypothesis stated in the previous section is satisfied.

Assume also that dim Ω = dim Δ (or $m = l - 1$), so that we are focusing precisely on those cases which are not included in the results of Allen [2, 4] or Jordan [27]. Note that the case of just as many parameters (describing the set of states of the world) as (relative) prices has some important and natural economic interpretations. For instance, suppose that the uncertainty concerns the prices of assets or commodities during the next period.

The following technical assumption is also necessary:

Uniform Boundedness Hypothesis. For all $\omega \in \Omega$ and almost all $\alpha \in A$, the derivatives $D_\omega p_\alpha(\omega)$ of agents' forecasts $p_\alpha: \Omega \to \Delta$ are uniformly bounded.

Theorem 1. *Given $q \in \Delta$, for almost all $\alpha \in A$, there are open neighborhoods $N_\alpha(q)$ of Δ such that $EU_\alpha(x; p_\alpha, q')$ is a C^1 function of q' on*

$N_\alpha(q)$ for all $x \in \mathbb{R}^l_{++}$. For these choices, conditional expected utilities are C^2 on the commodity space $\mathbb{R}^l_{++}$.

Proof. Think of $\omega \in \Omega$ as a generalized time parameter and p_α as a sample path of a generalized stochastic process on the probability triple $(A, \mathscr{B}(A), \lambda/\lambda(A))$ which takes values in $\bar{\Delta}$. A theorem of Bulinskaya ([14]; see also [17, p. 76]) states the following:

(i) Let u be fixed. If the one-dimensional density $f_t(x)$ of the process $\xi(t)$ is bounded in x and in $0 \leqslant t \leqslant 1$, and if $\xi(t)$ has, with probability one, a continuous sample derivative $\xi'(t)$, then the probability is zero that $\xi'(t) = 0$, $\xi(t) = u$ simultaneously, for any point t in $0 \leqslant t \leqslant 1$. In particular, there is zero probability of $\xi(t)$ being tangential to the level u anywhere in $0 \leqslant t \leqslant 1$.

(ii) Under the conditions of (i), the number of times $\xi(t) = u$ in $0 \leqslant t \leqslant 1$ is finite with probability one.

If $\Omega = \bar{\Delta} = [0, 1]$ (i.e., if $m = 1$ and $l = 2$), interpret the realization of the stochastic process (or sample path) as p_α, $t \in [0, 1]$ as $\omega \in \Omega = [0, 1]$, and u as q. Note that the boundedness of one-dimensional densities is assured by the Dispersion Hypothesis, If $l - 1 = \dim \Delta = \dim \Omega = m > 1$, the extension of Bulinskaya's Theorem to random fields given by Allen [7, Theorems 4.1 and 4.3], which uses the Uniform Boundedness Hypothesis, states that, under the Dispersion Hypothesis, for any $q \in \Delta$, except possibly for α in some null set (which depends on q), there is no $\omega \in \Omega$ for which $p_\alpha(\omega) = q$ and $\det D_\omega p_\alpha(\omega) = 0$. Moreover, for almost every $\alpha \in A$, there are only finitely many $\omega \in \Omega$ for which $p_\alpha(\omega) = q$. Eliminate this null set (depending on q) of $\alpha \in A$ where it is not the case that $p_\alpha(\cdot) \pitchfork \{q\}$. Call the remaining set of "good" agents $A(q)$. For each $\alpha \in A(q)$, q is a regular value of the smooth function p_α, or $p_\alpha \pitchfork \{q\}$. Since Ω is compact, $p_\alpha^{-1}(q)$ is a finite set whenever $\alpha \in A(q)$. (See [33, p. 8].)

It follows by a standard argument from the Implicit Function Theorem that for each $\alpha \in A(q)$ there is an open neighborhood $N(q)$ in $\bar{\Delta}$ and finitely many smooth functions $g_1, \ldots, g_{n(\alpha,q)}$ with

$$g_j : N_\alpha(q) \to \Omega$$

such that $p_\alpha^{-1}(q') = \{g_1(q'), \ldots, g_{n(\alpha,q)}(q')\}$ for all $q' \in N_\alpha(q)$. In particular, $\# p_\alpha^{-1}(\cdot) = n(\alpha, q)$ is constant on $N_\alpha(q)$. For $\alpha \in A(q)$, $q' \in N_\alpha(q)$, and $x \in \mathbb{R}^l_{++}$ define (recall that g is the density for μ)

$$EU_\alpha(x; p_\alpha, q') = \frac{\sum_{j=1}^{n(\alpha,q)} u_\alpha(x, g_j(q'))\, g(g_j(q'))|\det D_\omega p_\alpha(\omega)|^{-1}}{\sum_{j=1}^{n(\alpha,q)} g(g_j(q'))|\det D_\omega p_\alpha(\omega)|^{-1}}.$$

Note that, for α and q' as above, these expressions define α's conditional expected utility given $\{\omega \in \Omega \mid p_\alpha(\omega) = q'\}$. By interchanging limits, derivatives, and finite sums, it can be seen that $EU_\alpha(\cdot;\, p_\alpha,\, q')$ is C^2 on $\mathbb{R}^l_{++}$ for α and q' as above. Furthermore, for $\alpha \in A(q)$ and all $x \in \mathbb{R}^l_{++}$, $EU_\alpha(x;\, p_\alpha,\, \cdot)$ is C^1 on $N_\alpha(q)$. ∎

Remarks. Recall that $p_\alpha \pitchfork \{q\}$ means that whenever $\omega \in \Omega$ is such that $p_\alpha(\omega) = q$, then the $(l-1)\times(l-1)$ matrix $D_\omega p_\alpha(\omega)$ has full rank, or $\det D_\omega p_\alpha(\omega) \neq 0$. Transversality plays a key role in the argument because it is precisely where $D_\omega p_\alpha(\omega)$ drops rank that α's conditional expected utility, and hence demand, becomes (in general) discontinuous as a function of the price vector. By "spreading out" such discontinuities, so that for each $q \in \Delta$ they affect only a subset which has measure zero of agents in A, we can remove them by aggregation.

THEOREM 2. *For fixed $q \in \Delta$, there is a subset of imperfectly informed agents $A(q)$ of full measure, such that for each $\alpha \in A(q)$, there is an open neighborhood $N_\alpha(q)$, of q in Δ such that α's demand*

$$x_\alpha : N_\alpha(q) \to \mathbb{R}^l_{++}$$

and α's excess demand

$$z_\alpha : N_\alpha(q) \to \mathbb{R}^l$$

$$z_\alpha(q') = x_\alpha(q') - e_\alpha$$

are C^1 functions on $N_\alpha(q)$.

Proof. By inspection of the formula for $EU_\alpha(x; p_\alpha, q')$ for $\alpha \in A$ and $q' \in N_\alpha(q)$, it can be seen that these conditional expected utilities satisfy strict (differentiable) monotonicity, strict (differentiable) concavity, and the boundary condition. (Details are analogous to the proof of Lemma 4.1 in [1].) Furthermore, they are C^2 functions of the commodity bundle and C^1 functions of the parameter q'. Hence demand is a C^1 function when restricted to the set $N_\alpha(q) \cap \Delta$. This is true also for excess demands, which are just equal to the demand functions minus the constant initial endowment vector. ∎

COROLLARY 1. *Fix $q \in \Delta$, and choose $N(<\infty)$ agents randomly from the probability space $(A, \mathscr{B}(A), \lambda)$. Then, with probability one, there is an open neighborhood $U(q)$ of q in Δ such that their total demand is a C^1 function on $U(q)$.*

Proof. With probability one, all N agents are members of $A(q)$. Take

$U(q)=\bigcap_{i=1}^{N} N_{\alpha_i}(q)$ for the open neighborhood of q. On this set, each agent's demand is C^1 by the theorem. ∎

THEOREM 3. *The aggregate demand of the imperfectly informed agents*

$$X_A: \Delta \to \mathbb{R}^l_{++}$$

$$X_A(q)=\int_A x_\alpha(q)\, d\alpha$$

and their aggregate excess demand

$$Z_A: \Delta \to \mathbb{R}^l$$

$$Z_A(q)=\int_A z_\alpha(q)\, d\alpha = X_A(q)-\int_A e_\alpha\, d\alpha$$

are continuous functions.

Proof. Fix $q \in \Delta$, and let $\{q_n\}$ be a sequence in Δ converging to q. For continuity, I need to show that $X_A(q_n) \to X_A(q)$. Let K be a compact subset of Δ containing q and all of the q_n. For any $q' \in K$ and all $\alpha \in A$, the budget sets $\{x \in \mathbb{R}^l_{++} \mid q' \cdot x \leqslant q' \cdot e_\alpha\}$ are uniformly bounded. By definition, demands must lie in budget sets, so that this gives a uniform bound for demand vectors for all $\alpha \in A$, and all q_n and q. By the above, for almost every $\alpha \in A$, $x_\alpha(q_n) \to x_\alpha(q)$ as $n \to \infty$ (pointwise in $\alpha \in N(\alpha)$). Hence by dominated convergence,

$$\lim_{n \to \infty} \int_A x_\alpha(q_n)\, d\alpha = \int_A \lim_{n \to \infty} x_\alpha(q_n)\, d\alpha = \int_A x_\alpha(q)\, d\alpha$$

or

$$\lim_{q_n \to q} X_A(q_n) = X_A(q).$$

Similarly, $\lim_{q_n \to q} Z_A(q_n) = Z_A(q)$, since aggregate excess demand differs from aggregate demand by the finite constant $\int_A e_\alpha d\alpha$ which is independent of q. ∎

THEOREM 4. *For each $\omega \in \Omega$, aggregate excess demand at ω, $Z_\omega: \Delta \to \mathbb{R}^l$, given by*

$$\begin{aligned} Z_\omega(q) &= \int_A x_\alpha(q)\, d\alpha - \int_A e_\alpha d\alpha + \int_B x_\beta(q, \omega)\, d\beta - \int_B e_\beta(\omega)\, d\beta \\ &= Z_A(q) + Z_B(q, \omega) = Z(q, \omega) \end{aligned}$$

is continuous on Δ, satisfies Walras' Law, and obeys the boundary condition that if $q_n \to q \in \partial\Delta$, then the sequence $\{\|Z(q_n, \omega)\|\}$ is unbounded.

Proof. By Theorem 3, $Z_A(\cdot)$ is continuous on Δ. To see that $Z_B(\cdot, \omega)$ is continuous for each $\omega \in \Omega$, note that the individual demands for informed agents $x_\beta(q, \omega)$ were formed by maximizing the utilities $v_\beta(x, \omega)$ over the budget sets $\{x \in \mathbb{R}^l_{++} \mid q \cdot x \leqslant q \cdot e_\beta(\omega)\}$, so that, in fact, for each $\beta \in B$, $x_\beta(\cdot, \cdot)$ is C^1 in both variables. (Recall that $e_\beta(\cdot)$ was C^1 and v_β was C^1 on Ω and C^2 on $\mathbb{R}^l_{++}$ and that, furthermore, for each $\omega \in \Omega$, $v_\beta(\cdot, \omega)$ has no critical point and represents preferences which have indifference surfaces with nonvanishing Gaussian curvature.) Now let $q_n \to q$. By uniform boundedness of initial endowments, budget sets, and hence demands, are uniformly bounded for all $\beta \in B$ and all q_n and q. Use dominated convergence for each $\omega \in \Omega$ to conclude that

$$\lim_{q_n \to q} X_B(q_n, \omega) = \lim_{n \to \infty} \int x_\beta(q_n, \omega)\, d\beta = \int \lim_{n \to \infty} x_\beta(q_n, \omega)\, d\beta$$

$$= \int x_\beta(q, \omega)\, d\beta = X_B(q, \omega),$$

which says that $X_B(\cdot, \omega)$ is continuous on Δ. For any ω, $\int_B e_\beta(\omega)\, d\beta$ is finite. Thus, for each $\omega \in \Omega$,

$$\lim_{q_n \to q} Z_B(q_n, \omega) = \lim_{q_n \to q} X_B(q_n, \omega) - \int_B e_\beta(\omega)\, d\beta$$

$$= X_B(q, \omega) - \int_B e_\beta(\omega)\, d\beta = Z_B(q, \omega),$$

which shows that $Z_B(\cdot, \omega)$ is continuous on Δ. Hence $Z(\cdot, \omega) = Z_A(\cdot) + Z_B(\cdot, \omega)$ is a continuous function on Δ, for all $\omega \in \Omega$.

To see that $q \cdot Z(q, \omega) = 0$, $\forall \omega \in \Omega$, note that the individual excess demands satisfy Walras' Law:

$$q \cdot z_\alpha(q) = 0 \qquad \forall \alpha \in A$$

$$q \cdot z_\beta(q, \omega) = 0 \qquad \forall \beta \in B, \forall \omega \in \Omega.$$

Hence Walras' Law must also hold for their integral.

Finally, to prove that the boundary condition is satisfied, we shall again note that it is fulfilled for each agent. By Fatou's Lemma and continuity, for each commodity $j = 1, \ldots, l$

$$\begin{aligned}
&\int_A \liminf_{n\to\infty} x_\alpha^j(q_n)\,d\alpha + \int_B \liminf_{n\to\infty} x_\beta^j(q_n,\omega)\,d\beta \\
&\quad\leqslant \liminf_{n\to\infty} \int_A x_\alpha^j(q_n)\,d\alpha + \liminf_{n\to\infty} \int_B x_\beta^j(q_n,\omega)\,d\beta \\
&\quad\leqslant \liminf_{n\to\infty} \left[\int_A x_\alpha^j(q_n)\,d\alpha + \int_B x_\beta^j(q_n,\omega)\,d\beta \right] \\
&\quad\leqslant \liminf_{n\to\infty} \left[X_A^j(q_n) + X_B^j(q_n,\omega) \right] \\
&\quad\leqslant \lim_{n\to\infty} X^j(q_n,\omega) = X^j(q,\omega).
\end{aligned}$$

Since for each α, β, and ω, $\|x_\alpha(q_n)\| \to \infty$ and $\|x_\beta(q_n,\omega)\| \to \infty$, for each $\omega \in \Omega$, there is some commodity j and there are subsets $A^j \subset A$ and $B^{j\omega} \subset B$ with either $\lambda(A^j) > 0$ or $\lambda(B^{j\omega}) > 0$ such that

$$\begin{aligned}
x_\alpha^j(q_n) &\to \infty \text{ as } n \to \infty \qquad \forall \alpha \in A^j \\
x_\beta^j(q_n,\omega) &\to \infty \text{ as } n \to \infty \qquad \forall \beta \in B^{j\omega}.
\end{aligned}$$

Hence $X^j(q_n, \omega) \geqslant \int_{A^j} x_\alpha^j(q_n)\,d\alpha + \int_{B^{j\omega}} x_\beta^j(q_n,\omega)\,d\beta$ (because demands are bounded from below by 0—the consumption sets are $\mathbb{R}^l_{++}$). Now let $n \to \infty$ to conclude that, for each $\omega \in \Omega$, there is j $(1 \leqslant j \leqslant l)$ such that $X^j(q_n,\omega) \to \infty$. By definition, $Z^j(q_n,\omega) = X^j(q_n,\omega) - \int_A e_\alpha^j\,d\alpha - \int_B e_\beta^j(\omega)\,d\beta$ where the aggregate initial endowments are uniformly bounded (in α, β, and ω). Hence, for each ω, there is j such that $Z^j(q_n,\omega) \to \infty$ as $n \to \infty$. This proves that $\|Z(q_n,\omega)\| \to \infty$ as $n \to \infty$. ∎

THEOREM 5. *There is a market-clearing price function, i.e., there is*

$$\bar{p}: \Omega \to \Delta$$

such that, for all $\omega \in \Omega$, $Z(\bar{p}(\omega),\omega) = 0$.

Proof. Fix $\omega \in \Omega$ arbitrarily. $Z(\cdot,\omega)$ is continuous on Δ, satisfies Walras' Law, and the boundary condition. By [19, p. 78], these conditions imply that there is a $\bar{p}(\omega) \in \Delta$ such that $Z(\bar{p}(\omega),\omega) = 0$. ∎

4. EXISTENCE OF APPROXIMATELY RATIONAL EXPECTATIONS EQUILIBRIA

From the preceding results and assumptions, we know that, given any distribution of endowments and preferences for the informed agents and any distribution of utilities and forecasts for which the distribution of forecasts is dispersed, excess demand depends continuously on the price vector. Hence, there is a market-clearing price correspondence $\bar{p}: \Omega \to \Delta$. However, we have not shown any relationship between $\bar{p}$ and the forecasts of uninformed agents. Of course, if $\bar{p}$ were a C^1 function and if for every uninformed agent α, $p_\alpha = \bar{p}$, then $\bar{p}$ would form a rational expectations equilibrium for the economy. Unfortunately, such a distribution of forecasts is not dispersed—it is the atomic measure $\delta_{\bar{p}}$. If we were to perturb this disribution slightly in an appropriate way, we could have dispersed distributions of forecasts which are arbitrarily close to $\delta_{\bar{p}}$—that is, forecasts which are approximately correct. This is what is meant by the phrase "approximately rational expectations equilibrium."

Formally, say that $\bar{p}$ is an *ε-rational expectations equilibrium* if $\bar{p}$ is a market-clearing C^1 price function and

$$\max_{\omega \in \Omega} [\| p_\alpha(\omega) - \bar{p}(\omega)\| + \| Dp_\alpha(\omega) - D\bar{p}(\omega)\|] < \varepsilon \quad \text{for almost every } \alpha \in A.$$

This means that, except for a null set of agents, everyone's forecast is within ε of being correct for all $\omega \in \Omega$, and furthermore, the derivatives which reflect the dependence of prices on states of the world are also within ε of being completely accurate. Note that if $\bar{p}$ is a strict rational expectations equilibrium (in the sense of [2, 35]), then it is also an ε-rational expectations equilibrium for all $\varepsilon \geqslant 0$.

As a first step, we will make some additional assumptions.

Assumption A1 (Compact support). The distribution of informed agents' state-dependent utilities $v_\beta \in C^1(\Omega, C^2(\mathbb{R}^l_{++}, \mathbb{R}))$ has compact support (for the topology of C^2 uniform convergence on compact subsets of $\mathbb{R}^l_{++}$ and C^1 uniform convergence on the compact set Ω).

Proposition. *The informed agents' aggregate excess demand*

$$Z_B(q, \omega) = \int_B x^*_\beta(q; \omega)\, d\beta - \int_B e_\beta(\omega)\, d\beta$$

is a C^1 function from $\hat{\Delta} \times \Omega$ into $\mathbb{R}^l$, where $\hat{\Delta}$ denotes any compact subset of Δ.

Proof. By assumption, $e_\beta: \Omega \to \mathbb{R}^l_{++}$ is C^1 for each β and $e_\beta(\omega)$ and $De_\beta(\omega)$ are uniformly bounded in β and ω (except possibly for null set of β agents) by the compact support assumption. Also by Assumption A1, for almost all β, the state-dependent utilities $v_\beta(\cdot; \cdot)$ are contained in a compact subset of $C^1(\Omega, C^2(\mathbb{R}^l_{++}, \mathbb{R}))$. Hence, for almost all β, for all $\omega \in \Omega$, the $v_\beta(\cdot; \omega)$ lie in a compact subset of $C^2(\mathbb{R}^l_{++}, \mathbb{R})$. Hence, by [1], for almost all β and all ω, the demands $x^*_\beta(\cdot;\omega)$ lie in a compact subset of $C^1(\Delta, \mathbb{R}^l_{++})$.

To show smoothness of aggregate excess demand, we need to differentiate under the integral sign. For fixed $q \in \Delta$,

$$D_\omega Z_\beta(q, \omega) = D_\omega \int_B x^*_\beta(q;\omega)\, d\beta + D_\omega \int_\omega e_\beta(\omega)\, d\beta$$

$$= \int_B D_\omega x^*_\beta(q;\omega)\, d\beta + \int D_\omega e_\beta(\omega)\, d\beta$$

and for fixed $\omega \in \Omega$,

$$D_q Z_B(q, \omega) = D_q \int_B x^*_\beta(q;\omega)\, d\beta$$

$$= \int_B D_q x^*_\beta(q;\omega)\, d\beta.$$

(See [31, pp. 282, 375].)
On $\hat{\Delta} \times \Omega$, everything is uniformly bounded a.e. on B. Hence these derivatives are continuous by the dominated convergence theorem. ∎

Assumption A2 (Independence). Among uninformed agents, the distribution of state-dependent utility functions, the distribution of initial endowments, and the distribution of forecasts are independent.

Assumption A3 (Log-linear utilities for uninformed agents). For almost all $\alpha \in A$,

$$u_\alpha(x, \omega) = \sum_{j=1}^{l} f^j_\alpha(\omega) \log x_j$$

for some function $f: A \times \Omega \to \Delta(\gamma)$, where $\Delta(\gamma) = \{(y_1, ..., y_l) \in \mathbb{R}^l_{++} \mid \sum_{j=1}^l y_j = 1 \text{ and } \gamma \leqslant y_j \leqslant 1-\gamma,\ j = 1, ..., l\}$, which is measurable in α and C^2 in the parameter $\omega \in \Omega$.

Assumption A4 (Homogeneity of uninformed agents' preferences). Among uninformed agents, the distribution of utility functions associated with a given state of the world is independent of the state. That is, the "image measures" $\lambda \circ (f(\omega))^{-1}$ are independent of $\omega \in \Omega$.

Assumption A5 (Informed agents' preferences). For almost all $\beta \in B$ and all $\omega \in \Omega$, $v_\beta(\cdot, \omega)$ can be expressed as a strictly increasing transformation of some log-linear utility function (which depends on β and ω).

Remark. The purpose of Assumption A5 is to ensure that the market-clearing price correspondence $\bar{p}$ is a C^1 function. Sufficiently slight perturbations of the $v_\beta(\cdot, \omega)$ will not destroy this property. Alternatively, we could have assumed that, regardless of the p_α functions, the economy is regular and satisfies one of the properties (gross substitutes, dominant diagonal, etc.) which suffice to ensure uniqueness of competitive equilibrium in economies without uncertainty.

THEOREM 6. *Under all of the above assumptions, for any $\varepsilon > 0$ an ε-rational expectations equilibrium exists.*

Proof. For the original economy, with uninformed agents' forecasts p_α, there is an equilibrium price function $\bar{p}$: $\Omega \to \Delta$ which is C^1 by A3 and A5. To describe the resulting revisions of forecasts, let $r \in (0, 1)$. Think of r as describing how much weight agents place on the current observation. Define the forecasts revised (with respect to r) by $\bar{p}$ as, for each $\alpha \in A$ and $\omega \in \Omega$

$$p_\alpha^1(\omega) = r\bar{p}(\omega) + (1 - r)\, p_\alpha(\omega).$$

Note that p_α^1 is a C^1 function from Ω to Δ and, furthermore, that the distribution of p_α^1 satisfies the Dispersion Hypothesis. Hence, there is a market-clearing price function $\bar{p}^1$ for the economy in which uninformed agents have forecasts p_α^1. In fact, the crucial property which makes this an "easy" convergence theorem is that, for all $\omega \in \Omega$, $\bar{p}^1(\omega) = \bar{p}(\omega)$. To see this, note that, by A2, A3, and A4, for any given $q \in \Delta$, the demand of an uninformed agent α for commodity k is (except for a null subset of A)

$$x_\alpha^k(q) = \frac{a_\alpha \cdot q \sum_{j=1}^{n(\alpha,q)} f_\alpha(g_j(q))\, g(g_j(q)) |\det D_\omega\, p_\alpha(\omega)|^{-1}}{q^k \sum_{j=1}^{n(\alpha,q)} g(g_j(q)) |\det D_\omega\, p_\alpha(\omega)|^{-1}}$$

where, for almost all $\alpha \in A$, $\{\omega \mid p_\alpha(\omega) = q\}$ equals the finite subset $\{g_1(q), \ldots, g_{n(\alpha,q)}(q)\}$, as in the proof of Theorem 1. Let $Ee = \int e_\alpha \, d\alpha$ and write Ef for the vector $(1/\lambda(A)) \int f_\alpha(\omega)\, d\alpha$. The integral exists (the $f_\alpha(\omega)$ are bounded, positive and measurable in α), is independent of ω, and is strictly between zero and one (by A3). Since the distributions of p_α, e_α, and f_α are independent, aggregate demand of the uninformed agents is given by

$$X_A(q) = \int_A x_\alpha(q)\, d\alpha = q \cdot Ee \left(\frac{Ef^1}{q^1}, \ldots, \frac{Ef^l}{q^l}\right) \in \mathbb{R}^l_{++}.$$

$X_A(q)$ is a C^1 function on Δ and is independent of the forecasts p_α. Aggregate excess demand is $Z(q; \omega) = X_A(q) - Ee + Z_B(q, \omega)$, which is also independent of the p_α. This says that the market-clearing price function $\bar{p}$ is independent of the p_α. A similar argument shows that $Z^1(q;\omega)$—defined with p_α^1 in place of p_α—is independent of the p_α^1, since the definition of p_α^1 forces independence of the distributions of p_α^1 and f_α. In fact,

$$Z^1(q;\omega) = X_A(q) - Ee + Z_B(q,\omega) = Z(q,\omega)$$

which forces $\bar{p}(\omega) = \bar{p}^1(\omega)$ for all $\omega \in \Omega$. Defining $p_\alpha^2 = r\bar{p}^1 + (1-r)\, p_\alpha^1$ gives $\bar{p}^2 = \bar{p}^1 = \bar{p}$. Repeating the argument results in a constant sequence of market clearing price functions and a sequence of forecasts which satisfies all of our assumptions, including dispersion, and which "converges" to $\bar{p}$. We have, for each $\alpha \in A$,

$$\begin{aligned}
p_\alpha^n &= r\bar{p}^{n-1} + (1-r)\, p_\alpha^{n-1} \\
p_\alpha^n &= r\bar{p}^{n-1} + (1-r)[r\bar{p}^{n-2} + (1-r)\, p_\alpha^{n-2}] \\
&= r\bar{p} + (1-r)\, r\bar{p} + (1-r)^2\, [r\bar{p} + (1-r)\, p_\alpha^{n-3}] \\
&= (1-r)^n\, p_\alpha + [1-(1-r)^n]\, \bar{p}.
\end{aligned}$$

Hence $\| p_\alpha^n - \bar{p} \| \leqslant (1-r)^n \, \| p_\alpha - \bar{p} \|$ for all $\alpha \in A$. By the Uniform Boundedness Hypothesis and the fact that all forecasts and price functions take values in Δ, there is a (finite) number B such that $\| p_\alpha \| = \max_{\omega \in \Omega} [\| p_\alpha(\omega)\| + \| Dp_\alpha(\omega)\|] < B$ for almost all $\alpha \in A$. Hence

$$\| p_\alpha^n - \bar{p} \| \leqslant (1-r)^n\, B$$

for almost every $\alpha \in A$. Since $0 < r < 1$, $(1-r)^n \to 0$ as $n \to \infty$. Hence, taking n large enough so that $(1-r)^n\, B < \varepsilon$ gives an ε-rational expectations equilibrium, and completes the proof. ∎

COROLLARY 2. *As $\varepsilon \to 0$, the distributions of revised forecasts of uninformed agents converge weakly (as probability measures on $C^1(\Omega, \Delta)$) to the distribution which gives probability 1 to $\bar{p}$.*

COROLLARY 3. *As $\varepsilon \to 0$, the forecast representations*

$$p^n : A \to C^1(\Omega, \bar{\Delta}),$$

as measurable functions, converge in $\mathscr{L}^1(A, \mathscr{B}(A), \lambda, C^1(\Omega, \bar{\Delta}))$ to the constant function

$$\bar{p}.: A \to C^1(\Omega, \bar{\Delta})$$

defined by $\bar{p}_\alpha = \bar{p}$.

Proof. We need to examine

$$\int_A \| p_\alpha^n - \bar{p}_\alpha \| \, d\alpha = \int_A \max_{\omega \in \Omega} [\| p_\alpha^n(\omega) - \bar{p}_\alpha(\omega) \| + \| Dp_\alpha^n(\omega) - D\bar{p}_\alpha(\omega) \|] \, d\alpha$$

$$\leqslant \int_A (1-r)^n B \, d\alpha = \lambda(A) B(1-r)^n.$$

The right-hand side converges to zero as n becomes large, thereby completing the proof. ▮

Remark. Under our assumptions, the aggregate demand of uninformed agents is independent of their forecasts, and thus, of the number of revisions which have occurred. However, this is not true for individual agents. It could be the case that each agent is truly "learning" in the sense that, with the use of revised forecasts, his average ex post state-dependent utility (of consuming his demand) is increased.

5. Convergence

Examination of the proof of Theorem 6 suggests that convergence to rational expectations occurs as traders learn, since as the number of steps t becomes large, the revised forecasts p_α^t get arbitrarily close to the actual market clearing price function $\bar{p}^t$. The problem in this interpretation is that the sequencing does not seem to make sense as economic optimizing behavior in a competitive market.[3] Implicit in the proof was the notion that all agents learn the entire market equilibrium relation $\bar{p}^t$ at the end of step t and then simultaneously revise their beliefs about this relationship for all states of the world. Yet, before knowing the whole equilibrium price function, each trader obtains additional information about market-clearing prices for some states—for instance, the first state of the world to be realized after the previous revision. Immediate use of this information generally leads to higher utility for agents who behave in this way.

Hence, for an acceptable economic story of real-time dynamic convergence, all agents should update their forecasts each time a state of the world is realized. The explicit sequencing of such a learning process works

[3] I wish to thank Dave Cass for some helpful comments on this point that he made during a theory workshop at Penn. At that seminar, I interpreted the proof of an early version of Theorem 6 as a simple dynamics of forecast revision. His objections and remarks stimulated this section; he should not be held responsible for the outcome.

as follows: At date t, each agent determines his demand based on his current forecast p_α^t and the observed prevailing market price vector $\bar{p}^t(\omega^t)$. After trading occurs, the state ω^t of the world at time t is revealed to the agent. The information acquired in period t is thus that state ω^t has $\bar{p}^t(\omega^t)$ as its equilibrium price vector. This information should be incorporated into the agent's forecast before he enters the market at date $t+1$. If it is also known that the market-clearing price functions $\bar{p}^t$ are continuous, as they are in our model, then market-clearing prices at all states $\omega \in \Omega$ which are close to ω^t should be close to the price vector $\bar{p}^t(\omega^t)$. Revision of continuous forecast functions in a neighborhood of ω_t is appropriate. This "real-time" learning process and its convergence to approximately rational expectations equilibria is formalized in he following result:

THEOREM 7. *Assume that the economy satisfies A1–A5 and the dispersion hypothesis. Then, for "real-time dynamics," as $t \to \infty$, $p_\alpha^t \to \bar{p}$ C^0 uniformly on Ω and almost surely uniformly in $\alpha \in A$ with probability one.*

Remark 1. More specifically, the dynamics is defined by (where $\omega^t \in \Omega$ is realized at date t)

$$p_\alpha^{t+1}(\omega) = r_\alpha^t(\omega, \omega^t)\,\bar{p}(\omega^t) + (1 - r_\alpha^t(\omega, \omega^t))\,p_\alpha^t(\omega)$$

where the parameter mapping (for $0 < \hat{\gamma} \leqslant \gamma < 1$)

$$r^t : A \to C^0(\Omega \times \Omega, [0, \gamma])$$

is measurable and satisfies, $\forall \omega$, $\bar{\omega} \in \Omega$, $\forall \alpha \in A$, and $\forall t = 1, 2, 3, \ldots$,

$$r_\alpha^t(\omega, \bar{\omega}) > 0 \Leftrightarrow \| \omega - \bar{\omega} \| < 1/t$$

$$\| \omega - \bar{\omega} \| < 1/(t+1) \Rightarrow r_\alpha^t(\omega, \bar{\omega}) > \hat{\gamma}$$

The scalar $r_\alpha^t(\omega, \bar{\omega})$ tells how much agent α revises (at t) his forecast of the price associated with state of the world ω given that the realization at time t is $\bar{\omega} \in \Omega$.

Remark 2. An alternative way of stating the conclusion is as follows: $\Pr\{\{\omega^t\}_{t=1}^\infty \mid \forall \varepsilon > 0, \exists T$ such that $t \geqslant T \Rightarrow \| p_\alpha^t - \bar{p} \|_{C^0} < \varepsilon$ for a.e. $\alpha \in A\} = 1$.

Proof. Under the assumptions, I have a unique $\bar{p} : \Omega \to \Delta$ which is C^1 and independent of the p_α^ts. I want to show that $\Pr\{\{\omega^t\}_{t=1}^\infty \mid p_\alpha^t \to \bar{p}$ C^0 uniformly on Ω, uniformly for a.e. α, as $t \to \infty\} = 1$ where the adjustment weights that tell how p_α^t is revised are appropriately chosen as assumed in the statement of the theorem. To simplify, assume that μ is the uniform distribution on $(\Omega, \mathscr{F})$.

Fix α and suppose that $\omega^1, \omega^2, \ldots,$ is the sequence of realizations of the random variable. At the end of the date t, α can observe ω^t and $\bar{p}(\omega^t)$. Then, at any fixed $\hat{\omega} \in \Omega$, agent α's forecasts are given by

$$\begin{aligned}
p_\alpha^1(\hat{\omega}) &= r_\alpha^1(\hat{\omega}, \omega^1)\,\bar{p}(\omega^1) + (1 - r_\alpha^1(\hat{\omega}, \omega^1))\,p_\alpha(\hat{\omega}) \\
p_\alpha^2(\hat{\omega}) &= r_\alpha^2(\hat{\omega}, \omega^2)\,\bar{p}(\omega^2) + (1 - r_\alpha^2(\hat{\omega}, \omega^2))\,p_\alpha(\hat{\omega}) \\
&= r_\alpha^2(\hat{\omega}, \omega^2)\,\bar{p}(\omega^2) + (1 - r_\alpha^2(\hat{\omega}, \omega^2)) \\
&\quad \times [r_\alpha^1(\hat{\omega}, \omega^1)\,\bar{p}(\omega^1) + (1 - r_\alpha^1(\hat{\omega}, \omega^1))\,p_\alpha^1(\hat{\omega})] \\
&= r_\alpha^2(\hat{\omega}, \omega^2)\,\bar{p}(\omega^2) + r_\alpha^1(\hat{\omega}, \omega^1)\,\bar{p}(\omega^1) \\
&\quad + p_\alpha(\hat{\omega}) - r_\alpha^1(\hat{\omega}, \omega^1)\,p_\alpha(\hat{\omega}) \\
&\quad - r_\alpha^2(\hat{\omega}, \omega^2)\,r_\alpha^1(\hat{\omega}, \omega^1)\,\bar{p}(\omega^1) \\
&\quad - r_\alpha^2(\hat{\omega}, \omega^2)\,p_\alpha(\hat{\omega}) \\
&\quad + r_\alpha^2(\hat{\omega}, \omega^2)\,r_\alpha^1(\hat{\omega}, \omega^1)\,p_\alpha(\hat{\omega}) \\
&= r_\alpha^2(\hat{\omega}, \omega^2)\,\bar{p}(\omega^2) + [r_\alpha^1(\hat{\omega}, \omega^1) \\
&\quad - r_\alpha^2(\hat{\omega}, \omega^2)\,r_\alpha^1(\hat{\omega}, \omega^1)]\,\bar{p}(\omega^1) \\
&\quad + [1 - r_\alpha^1(\hat{\omega}, \omega^1) - r_\alpha^2(\hat{\omega}, \omega^2) \\
&\quad + r_\alpha^2(\hat{\omega}, \omega^2)\,r_\alpha^1(\hat{\omega}, \omega^1)]\,p_\alpha(\hat{\omega}).
\end{aligned}$$

Let $a_t = r_\alpha^t(\hat{\omega}, \omega^t)$. Then

$$\begin{aligned}
p_\alpha^2(\hat{\omega}) &= a_2\,\bar{p}(\omega^2) + [a_1 - a_1 a_2]\,\bar{p}(\omega^1) \\
&\quad + [1 - a_1 - a_2 + a_1 a_2]\,p_\alpha(\hat{\omega}) \\
&= a_2\,\bar{p}(\omega^2) + a_1(1 - a_2)\,\bar{p}(\omega^1) \\
&\quad + (1 - a_1)(1 - a_2)\,p_\alpha(\hat{\omega}).
\end{aligned}$$

More generally, one can show that

$$p_\alpha^t(\hat{\omega}) = \prod_{s=1}^{t} (1 - a_s)\,p_\alpha(\hat{\omega}) + \sum_{s=1}^{t} b_{s,t}\,\bar{p}(\omega^s)$$

where $\prod_{s=1}^{t} (1 - a_s) + \sum_{s=1}^{t} b_{s,t} = 1$ for some $b_{s,t} \in [0, 1]$.

Pick $\varepsilon > 0$ arbitrarily. I claim that there is $T(\varepsilon)$, which also depends on the sequence $\{\omega^t\}$ of realizations of the random variable, such that, for almost every α, $\| p_\alpha^t(\hat{\omega}) - \bar{p}(\hat{\omega})\| < \varepsilon$, $\forall t \geqslant T(\varepsilon)$.

Let $T_1(\varepsilon)$ be such that $t \geqslant T_1(\varepsilon)$ and $\|\omega - \hat{\omega}\| < 1/t \Rightarrow \|\bar{p}(\omega) - \bar{p}(\hat{\omega})\| < \varepsilon/2$. Such a finite $T_1(\varepsilon)$ exists by the uniform continuity of $\bar{p}$ on the compact set Ω. Then $t \geqslant T_1(\varepsilon) \Rightarrow r_\alpha^t(\hat{\omega}, \omega) > 0$ only if

$\| \bar{p}(\hat{\omega}) - \bar{p}(\omega)\| < \varepsilon/2$, i.e., when summed over $s = T_1(\varepsilon)$ to $s = t$, $\sum b_{s,t}\, \bar{p}(\omega^s)$ is a nonnegative weighted sum with positive weight only on those $\bar{p}(\omega^s)$ which are within $\varepsilon/2$ of $\bar{p}(\hat{\omega})$. Hence, if we renumber so that $p_\alpha = p_\alpha^{T(\varepsilon)}$, then

$$p_\alpha^t(\hat{\omega}) = \left[\prod_{s=1}^{t} (1-a_s)\right] p_\alpha(\hat{\omega}) + \left[1 - \prod_{s=1}^{t} (1-a_s)\right] \tilde{p}^s$$

where $\| \tilde{p}^s - \bar{p}(\hat{\omega})\| < \varepsilon/2$, since a convex combination $\tilde{p}^s$ which gives positive weight only to points in $[\bar{p}(\hat{\omega}) - \varepsilon/2, \bar{p}(\hat{\omega}) + \varepsilon/2]$ must also lie in the same interval. Hence

$$p_\alpha^t(\hat{\omega}) - \bar{p}(\hat{\omega}) = \left[\prod_{s=1}^{t} (1-a_s)\right] (p_\alpha(\hat{\omega}) - \tilde{p}^s) + \tilde{p}^s - \bar{p}(\hat{\omega})$$

$$\| p_\alpha^t(\hat{\omega}) - \bar{p}(\hat{\omega})\| \leqslant \left[\prod_{s=1}^{t} (1-a_s)\right] \cdot 1 + \| \tilde{p}^s - \bar{p}(\hat{\omega})\|$$

$$\leqslant \left[\prod_{s=1}^{t} (1-a_s)\right] + \varepsilon/2.$$

By assumption, $\exists \hat{\gamma} > 0$ such that $\| \omega - \bar{\omega} \| < 1/(t+1) \Rightarrow r_\alpha^t(\omega, \bar{\omega}) > \hat{\gamma}$, $\forall \alpha$. This says that $\| \omega^t - \hat{\omega} \| < 1/(t+1) \Rightarrow 1 - a_t < 1 - \hat{\gamma} < 1$. Write S_t for the event $\omega \in [\hat{\omega} - 1/(t+1),\ \hat{\omega} + 1/(t+1)]$. Then for all t, $\Pr\{\omega^t \in S^t\} = 2/(t+1)$, since we assumed that, in each period, the probability law on Ω is given by the uniform distribution. Note that the ω^t are independent random variables, $t = 1, 2, 3, \ldots$, and $\sum_{t=1}^{\infty} \Pr\{\omega^t \in S^t\} = \sum_{t=1}^{\infty} 2/(t+1) = 2 \sum_{t'=2}^{\infty} 1/t' = \infty$. Hence $\Pr\{\omega^t \in S^t$ infinitely often$\} = 1$ by the Borel Zero-One Law [32, p. 228]. Thus, there are infinitely many positive integers t for which $1 - a_t < 1 - \hat{\gamma} < 1$.

Choose $T_2(\varepsilon)$ such that $(1 - \hat{\gamma})^{T_2(\varepsilon)} < \varepsilon/2$. Then for $t \geqslant T_1(\varepsilon) + T_2(\varepsilon)$ we have $\| p_\alpha^t(\hat{\omega}) - \bar{p}(\hat{\omega})\| \leqslant \varepsilon$. Our choices of $T_1(\varepsilon)$ and $T_2(\varepsilon)$ are independent of α.

Now I need to show that for a given $\varepsilon > 0$, I can take $T_1(\varepsilon)$ and $T_2(\varepsilon)$ independent of $\hat{\omega} \in \Omega$. For $T_1(\varepsilon)$, it is clear by the uniform continuity of $\bar{p}$ on the compact set Ω. For $T_2(\varepsilon)$, note that if $T_2(\varepsilon; \hat{\omega})$ works for $\hat{\omega} \in \Omega$, then it also works for all $\hat{\hat{\omega}} \in S(T_2(\varepsilon); \hat{\omega}) = (\hat{\omega} - 1/(T_2(\varepsilon) + 1),\ \hat{\omega} + 1/(T_2(\varepsilon) + 1))$. Hence, by compactness, there is a finite collection of $S(T_2(\varepsilon); \hat{\omega})$ which cover Ω. Call them $S(T_2^1(\varepsilon); \hat{\omega}_1), \ldots, S(T_2^M(\varepsilon); \hat{\omega}_M)$. Let $T_2(\varepsilon; \Omega) = \max\{T_2^1(\varepsilon), \ldots, T_2^M(\varepsilon)\}$. Hence for $\varepsilon > 0$, with probability one there exists $T = T_1(\varepsilon) + T_2(\varepsilon; \Omega) < \infty$ such that $t \geqslant T \Rightarrow \max_{\omega \in \Omega} \| p_\alpha^t(\omega) - \bar{p}(\omega)\| < \varepsilon$ for every α. In other words $Pr\{\{\omega^t\}_{t=1}^{\infty} \mid \sup_\alpha \max_{\omega \in \Omega} \| p_\alpha^t(\omega) - \bar{p}(\omega)\|$ becomes arbitrarily small for sufficiently large $t\} = 1 = \Pr\{\{\omega^t\}_{t=1}^{\infty} \mid p_\alpha^t \to \bar{p}\ \ C^0$ uniformly on Ω as $t \to \infty$, uniformly on a set of α of full measure$\}$. ∎

Remark 1. If $\bar{p}$ is revealing—that is, if it is injective on Ω—then $\bar{p}$ forms a strict rational expectations equilibrium and the above argument shows that this simple "learning process" converges to it. In this case, approximately rational expectations equilibria converge monotonically to revealing strict rational expectations equilibria, as defined in [2, 35].

Remark 2. Convergence to rational expectations has recently been demonstrated by Blume and Easley, Bray, and Radner for some classes of parametric models. Blume and Easley [11] obtained convergence of a class of learning processes (which includes Bayesian revision of priors) when there are a finite number of models, one of which is correct. Both Bray [13] and Radner [36] examined linear models and the use of ordinary least squares estimation.

6. Concluding Remarks

We have shown that, in a simple model in which "information" is suitably dispersed, discontinuous use of it by individual agents does not lead to nonexistence of market-clearing prices in large economies. Moreover, an example shows that, at least under some quite strong assumptions, ε-rational expectations equilibria exist and converge as agents revise their forecasts.

Note that we permit both Ω and $\bar{\Delta}$—the set of parameters and the set of normalized price vectors—to equal the closed unit interval. This is one of the cases in which generic existence of rational expectations equilibria fails in finite economies [29]. In fact, the Jordan–Radner [29] counterexample features log-linear utilities and probability distributions having smooth density functions. Sections 4 and 5 show that existence of ε-rational expectations equilibria, for all $\varepsilon > 0$, and their convergence as $\varepsilon \to 0$, can hold for some cases in which (i.e., $\Omega = \bar{\Delta} = [0, 1]$) open counterexamples to the existence of strict rational expectations equilibria are known. In this case, the ε-rational expectations equilibria are not, in general, revealing, even though they tend to transmit a nontrivial amount of information.

An implication of the dispersion hypothesis is that no set of forecasts of the form $\{p_\alpha \mid p_\alpha(\omega) = q\}$ ever receives positive probability, for any fixed choice of $\omega \in \Omega$ and $q \in \Delta$. This condition is necessary, but not sufficient, for the dispersion hypothesis to be satisfied. This observation suggests that a test which could contradict (but not verify) the dispersion hypothesis is to examine the (marginal) distribution of $p_\alpha(\omega)$ on Δ for some values of $\omega \in \Omega$. If the distribution were concentrated at a finite number of points for any $\omega \in \Omega$, the dispersion hypothesis would be violated. If the empirical marginal distribution were approximately normal, uniform, etc., then it would not assign positive probability to any price vector in Δ.

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